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A *q*-Analogue of a Two-Dimensional Hydrogen Atom

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A new method for the *q*-analogue of a hydrogen atom is presented. By using the method, we construct a model of a two-dimensional *q*-hydrogen atom, and determine its energy, angular momentum, and state vectors. The method may be extended to cases of arbitrary dimension.

1. INTRODUCTION

In recent years, several authors (e.g., Kibler and Negadi, 1991; Gora, 1992; Song and Liao, 1992) have studied the *q*-analogue of a hydrogen atom. They gave the energy spectrum and the *q*-hydrogen atom, but did not discuss its angular momentum and its state vector.

A method used in previous work (Kibler and Negadi, 1991; Gora, 1992) for constructing a *q*-hydrogen atom is to use results on the *q*-harmonic oscillator and make the *q*-analogue for a relation between a hydrogen atom and a four-dimensional harmonic oscillator. The derivation and meaning of this relation are as follows: The use of the so-called KS transformation transforms the energy eigenequation of the hydrogen atom

$$\left(-\frac{\hbar^2}{2\mu}\Delta_x - \frac{Ze^2}{x}\right)\psi = E\psi \tag{1}$$

into the energy eigenequation of the four-dimensional harmonic oscillator

$$\left(-\frac{\hbar^2}{2\mu}\Delta_u - 4Eu^2\right)\psi = 4Ze^2\psi \tag{2}$$

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accompanied by a constraint condition (Kibler and Negadi, 1984), where x_i ($i = 1, 2, 3$) and u_α ($\alpha = 1, 2, 3, 4$) are the coordinates of the hydrogen atom and the four-dimensional harmonic oscillator respectively; $x = (x_i x_i)^{1/2}$, $u^2 = u_\alpha u_\alpha$. In this transformation, the same function ψ is used to represent simultaneously the energy eigenfunctions of the two quantum systems, and the energies of the two quantum systems do not correspond one to one. In making the q-analogue for the transformation between the two quantum systems, the hydrogen atom and the four-dimensional harmonic oscillator, one may give the energy of the q-hydrogen atom, but cannot determine its state vectors.

In this paper, we will present a new method for the q-analogue of a hydrogen atom and construct a model of the q-hydrogen atom of dimension two. We use also a relation between a hydrogen atom and a harmonic oscillator, but this relation is not the same as the one mentioned above.

Zeng *et al.* (1994a,b) used the $SU(1,1)$ algebra to give the simplest and most general algebraic relationship between a d-dimensional hydrogen atom and a D-dimensional harmonic oscillator, in which the energies of the two quantum systems correspond one to one. Making a q-analogue for this relationship is an excellent way to construct a q-hydrogen atom. This paper is based on Zeng *et al.* (1994a,b), but we restrict ourselves to the case of dimension 2. We work out not only the energies of the two-dimensional (2D) q-hydrogen atom, but also its angular momentum and state vectors.

Remainder of this paper is organized as follows. In Section 2, the $SU(1,1)$ algebra is used to establish a connection between a 2D hydrogen atom and a 2D harmonic oscillator, in which energies, angular momenta, and state vectors of the two quantum systems correspond one to one. In Section 3, we give a q-analogue for the 2D harmonic oscillator in which the Hamiltonian and angular momentum operator of the 2D q-harmonic oscillator are defined and the eigenvalues and common eigenstates of these operators are determined. Then we make a q-analogue for the relation between the 2D hydrogen atom and the 2D harmonic oscillator given in Section 2, and construct a model of a 2D q-hydrogen atom.

2. RELATIONSHIP BETWEEN THE 2D HYDROGEN ATOM AND 2D HARMONIC OSCILLATOR

We denote the Hamiltonian and the angular momentum operator of the 2D hydrogen atom by h and l_3, respectively, and their common eigenstates by $|n, m\rangle$; thus one has

$$h|n, m\rangle = e_n|n, m\rangle, \qquad l_3|n, m\rangle = m|n, m\rangle \tag{3}$$

where $e_n = -1/[2(n + 1/2)^2]$ (in units $\mu e^4/\hbar^2$) are energy eigenvalues of the 2D hydrogen atom, $n = 0, 1, 2, \ldots$; $m = 0, \pm 1, \pm 2, \ldots, \pm n$.

We denote the Hamiltonian and the angular momentum operator of the 2D harmonic oscillator by H and L_3, respectively, and their common eigenstates by $|N, M\rangle'$; thus, one has

$$H|N, M\rangle' = E_N|N, M\rangle', \qquad L_3|N, M\rangle' = M|N, M\rangle' \tag{4}$$

where $E_N = N + 1$ (in units $\hbar\omega$) are the energy eigenvalues of the 2D harmonic oscillator, $N = 0, 1, 2, \ldots$; $|M| = N, N - 2, \ldots, 1$ or 0 depending on odd N or even N, respectively.

We now apply the method presented in Zeng (1994a,b) to establish a relation between the 2D hydrogen atom and the 2D harmonic oscillator.

Using the coordinates of the 2D hydrogen atom and 2D harmonic oscillator x_i ($i = 1, 2$) and u_α ($\alpha = 1, 2$) (in the following, these are assumed to be dimensionless), we construct the operators

$$K_1 = \frac{1}{2}(x\Delta_x + x), \qquad K_2 = i\left(\frac{1}{2} + x_i\frac{\partial}{\partial x_i}\right), \qquad K_3 = -\frac{1}{2}(x\Delta_x - x) \tag{5}$$

and

$$K_1 = \frac{1}{4}(u^2 + \Delta_u), \qquad K_2 = \frac{i}{2}\left(1 + u_\alpha\frac{\partial}{\partial u_\alpha}\right), \qquad K_3 = \frac{1}{4}(u^2 - \Delta_u) \tag{6}$$

respectively, where $x = (x_i x_i)^{1/2}$, $\Delta_x = \partial^2/\partial x_i \partial x_i$ and $u = (u_\alpha u_\alpha)^{1/2}$, $\Delta_u = \partial^2/\partial u_\alpha \partial u_\alpha$. It is easy to prove that the two sets of operators satisfy all the commutation relations of the $SU(1,1)$ algebra, i.e., $[K_1, K_2] = -iK_3$, $[K_2, K_3] = iK_1$, and $[K_3, K_1] = iK_2$. Thus the $SU(1,1)$ algebra establishes a connection between the 2D hydrogen atom and 2D harmonic oscillator. By using this connection, all the problems with the transformation between the two quantum systems may be solved.

Because the operators expressed in (5) and (6) are all elements of the $SU(1, 1)$ algebra, we can establish the operator equations

$$x\Delta_x + x = \frac{1}{2}(u^2 + \Delta_u), \qquad x_i\frac{\partial}{\partial x_i} = \frac{1}{2}u_\alpha\frac{\partial}{\partial u_\alpha}, \qquad x\Delta_x - x = -\frac{1}{2}(u^2 - \Delta_u) \tag{7}$$

The first and third of these equations may be combined to give

$$x = \frac{1}{2}u^2, \qquad x\Delta_x = \frac{1}{2}\Delta_u \tag{8}$$

The coordinate transformation between the 2D hydrogen atom and the 2D harmonic oscillator may be determined by using the above operator equations. Its form is not naturally unique. The transformation we give is as follows:

$$x_1 = \frac{1}{2}(u_1^2 - u_2^2), \qquad x_2 = u_1 u_2 \tag{9}$$

In using the pole coordinates, we set $x_1 = x\cos\alpha$ and $x_2 = x\sin\alpha$ for the 2D hydrogen atom, and $u_1 = u\cos\theta$ and $u_2 = u\sin\theta$ for the 2D harmonic oscillator. Substituting these relations into (9), we find the relation between α and θ

$$\alpha = 2\theta \tag{10}$$

The angular momentum operators of the 2D hydrogen atom and the 2D harmonic oscillator are $l_3 = -i\,\partial/\partial\alpha$ and $L_3 = -i\,\partial/\partial\theta$, respectively. According to (10), one easily shows that there is a relation between L_3 and l_3

$$L_3 = 2l_3 \tag{11}$$

It is clear that an eigenstate of l_3 (or L_3) must be an eigenstate of L_3 (or l_3); moreover, there is a relation between the corresponding eigenvalues

$$M = 2m \tag{12}$$

where m and M are the eigenvalues of l_3 and L_3, respectively. Equation (12) shows that the angular momentum of the 2D harmonic oscillator connected to the 2D hydrogen atom is always even.

Now we study the relation between the energies of the two quantum systems. The Hamiltonian of the 2D hydrogen atom is $h = -\frac{1}{2}\Delta_x - 1/x$. Using (5), this expression may be reduced to

$$(K_1 + K_3)h = -\frac{1}{2}(K_1 - K_3) - 1 \tag{13}$$

Acting with the operator equation (13) on the state vector $|n, m\rangle$ and using (3), we obtain

$$\{-[\tfrac{1}{2} + e_n]K_1 + [\tfrac{1}{2} - e_n]K_3 - 1\}|n, m\rangle = 0 \tag{14}$$

Defining the function θ_n by

$$\cosh\theta_n = \frac{1 - 2e_n}{\sqrt{-8e_n}}, \qquad \sinh\theta_n = -\frac{1 + 2e_n}{\sqrt{-8e_n}} \tag{15}$$

and using the relation satisfied by the elements of the $SU(1, 1)$ algebra

$$e^{-iK_2\theta_n}K_3 e^{iK_2\theta_n} = K_3\cosh\theta_n + K_1\sinh\theta_n \tag{16}$$

then we can rewrite equation (14) as

$$\left\{K_3 - \frac{1}{\sqrt{-2e_n}}\right\} e^{iK_2\theta_n} |n, m\rangle = 0 \tag{17}$$

After the coordinate transformation is made, the operator K_3 in (17) may be replaced by the one expressed in (6). Noting that $K_3 = \frac{1}{2}H$, where H is the Hamiltonian of the 2D harmonic oscillator, we conclude that (17) may be considered as the energy eigenequation of the 2D harmonic oscillator; in other words, one may establish the map

$$|N, M\rangle' = e^{iK_2\theta_n}|n, m\rangle, \qquad E_N = \frac{2}{\sqrt{-2e_n}} \tag{18}$$

or

$$|n, m\rangle = e^{-iK_2\theta_n}|N, M\rangle', \qquad e_n = -\frac{2}{(E_N)^2} \tag{19}$$

Equations (18) and (19) are just the relations between the energies and state vectors of the two quantum systems.

It is well known that $e_n = -1/[2(n + \frac{1}{2})^2]$, $n = 0, 1, 2, \ldots$; $E_N = N + 1$, $N = 0, 1, 2, \ldots$. Substituting these into (19), one gets

$$N = 2n \tag{20}$$

which means that the energy level of the 2D harmonic oscillator connected to the 2D hydrogen atom is always even.

It is easy to show that the operator K_2 commutes with the operators l_3 and L_3, i.e., one has

$$[K_2, l_3] = [K_2, L_3] = 0 \tag{21}$$

From this fact, when $|N, M\rangle'$ on the right side of (19) is an eigenstate of L_3, then $|n, m\rangle$ on the left side of (19) must be an eigenstate of l_3.

In the following, we use particle number representation to express the state vectors of the 2D harmonic oscillator. Introducing the creation and annihilation operators $a_\alpha^\dagger$ and a_α by

$$a_\alpha^\dagger = \frac{1}{\sqrt{2}}\left(u_\alpha - \frac{\partial}{\partial u_\alpha}\right), \qquad a_\alpha = \frac{1}{\sqrt{2}}\left(u_\alpha + \frac{\partial}{\partial u_\alpha}\right) \tag{22}$$

which satisfy

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}, \qquad [a_\alpha, a_\beta] = 0, \qquad [a_\alpha^\dagger, a_\beta^\dagger] = 0 \tag{23}$$

then we can write the Hamiltonian and the angular momentum operator of the 2D harmonic oscillator as

$$H = \frac{1}{2} \sum_{\alpha=1,2} (a_\alpha^\dagger a_\alpha + a_\alpha a_\alpha^\dagger) \tag{24}$$

and

$$L_3 = -i(-a_1 a_2^\dagger + a_1^\dagger a_2) \tag{25}$$

respectively. One may use a representation in which both $a_1^\dagger a_1$ and $a_2^\dagger a_2$ are diagonalized; in this case, H is diagonalized also, but L_3 is not diagonalized.

In order to diagonalize simultaneously H and L_3, we introduce the operators a_+ and a_- by setting

$$a_1 = \frac{1}{\sqrt{2}} (a_+ + a_-), \qquad a_2 = \frac{i}{\sqrt{2}} (a_+ - a_-) \tag{26}$$

The commutation relations satisfied by $a_\pm$ and $a_\pm^\dagger$ are the same as (23), provided one replaces the indexes 1 and 2 by + and −, respectively. Using the transformation (26) reduces H and L_3 to

$$H = \tfrac{1}{2}(a_+^\dagger a_+ + a_+ a_+^\dagger + a_-^\dagger a_- + a_- a_-^\dagger) \tag{27}$$

and

$$L_3 = a_+^\dagger a_+ - a_-^\dagger a_- \tag{28}$$

respectively. Let $|m_+\rangle''$ and $|m_-\rangle''$ denote the eigenstates of $a_+^\dagger a_+$ and $a_-^\dagger a_-$ with the eigenvalues m_+ and m_-, respectively; then in the representation constructed by $|m_+, m_-\rangle'' = |m_+\rangle'' |m_-\rangle''$, both H and L_3 are diagonalized, and one has

$$H|m_+, m_-\rangle'' = (m_+ + m_- + 1)|m_+, m_-\rangle'' \tag{29}$$

$$L_3|m_+, m_-\rangle'' = (m_+ - m_-)|m_+, m_-\rangle'' \tag{30}$$

It is clear that one may express $|N, M\rangle'$ in terms of $|m_+, m_-\rangle''$, i.e., set $|N, M\rangle' = |m_+, m_-\rangle''$ accompanied by $N = m_+ + m_-$, $M = m_+ - m_-$. In this case, the formula (19) can be reduced to

$$|n, m\rangle = e^{-iK_2\theta_n}|m_+, m_-\rangle'' \tag{31}$$

where the operator K_2 takes the form

$$K_2 = -\frac{i}{2} (a_+^\dagger a_-^\dagger - a_+ a_-) \tag{32}$$

In (6), replacing u_α and $\partial/\partial u_\alpha$ by a_α and $a_\alpha^\dagger$ and then making the transformation (26), one obtains (32).

3. 2D q-HYDROGEN ATOM

Because we will construct a 2D q-hydrogen atom by means of a 2D q-harmonic oscillator, we now describe some results on the 2D q-harmonic oscillator. The creation and annihilation operators of the 2D q-harmonic oscillator $a_{q\alpha}^{\dagger}$ and $a_{q\alpha}$ (in the following, $\alpha = +, -$) may be defined as (Macfarlane, 1989; Biedenharn, 1989; Katriel and Solomon, 1991)

$$a_{q\alpha}^{\dagger} = a_{\alpha}^{\dagger}\left\{\frac{[M_\alpha + 1]}{M_\alpha + 1}\right\}^{1/2}, \qquad a_{q\alpha} = \left\{\frac{[M_\alpha + 1]}{M_\alpha + 1}\right\}^{1/2} a_\alpha \tag{33}$$

where M_α are the number operators, and $[x] = (q^{x} - q^{-x})/(q - q^{-1})$. These operators satisfy the relations

$$a_{q\alpha}^{\dagger}a_{q\alpha} = [M_\alpha], \qquad a_{q\alpha}a_{q\alpha}^{\dagger} = [M_\alpha + 1], \qquad a_{q\alpha}a_{q\alpha}^{\dagger} - qa_{q\alpha}^{\dagger}a_{q\alpha} = q^{-M_\alpha} \tag{34}$$

$$[a_{q\alpha}, a_{q\beta}] = [a_{q\alpha}^{\dagger}, a_{q\beta}^{\dagger}] = [M_\alpha, M_\beta] = 0 \tag{35}$$

The Hamiltonian and the angular momentum operator of the 2D q-harmonic oscillator may be defined as

$$H_q = \frac{1}{2}\sum_{\alpha=+,-}(a_{q\alpha}^{\dagger}a_{q\alpha} + a_{q\alpha}a_{q\alpha}^{\dagger}) = \frac{1}{2}\sum_{\alpha=+,-}([M_\alpha] + [M_\alpha + 1]) \tag{36}$$

and

$$L_{3q} = a_{q+}^{\dagger}a_{q+} - a_{q-}^{\dagger}a_{q-} = [M_+] - [M_-] \tag{37}$$

respectively. Equations (33)–(37) are just a q-generalization of the operators of the 2D harmonic oscillator.

The basis vectors spanning the representation of the deformed algebra (33)–(35) may be denoted by $|m_+, m_-\rangle_q''$. This is the eigenvector of the operator M_α with the eigenvalue m_α:

$$M_\alpha|m_+, m_-\rangle_q'' = m_\alpha|m_+, m_-\rangle_q'' \tag{38}$$

and satisfies

$$a_{q\alpha}|m_+, m_-\rangle_q'' = [m_\alpha]^{1/2}|m_+ - \delta_{\alpha+}, m_- - \delta_{\alpha-}\rangle_q'' \tag{39}$$

$$a_{q\alpha}^{\dagger}|m_+, m_-\rangle_q'' = [m_\alpha + 1]^{1/2}|m_+ + \delta_{\alpha+}, m_- + \delta_{\alpha-}\rangle_q'' \tag{40}$$

Obviously, the basis vector $|m_+, m_-\rangle_q''$ is also the common eigenvector of H_q and L_{3q}, and the corresponding eigenequations may be written as

$$H_q|m_+, m_-\rangle_q'' = E_{qN}|m_+, m_-\rangle_q'' \tag{41}$$

and

$$L_{3q}|m_+, m_-\rangle_q'' = M_q|m_+, m_-\rangle_q'' \tag{42}$$

respectively, where E_{qN} and M_q are the eigenvalues of H_q and L_{3q}, respectively, and

$$E_{qN} = \frac{1}{2} \sum_{\alpha=+,-} ([m_\alpha] + [m_\alpha + 1]) \tag{43}$$

$$M_q = [m_+] - [m_-] \tag{44}$$

Now let us describe our method for the q-analogue of a hydrogen atom. We have given the relationships between the energy, angular momentum, and state vector of the 2D hydrogen atom and those of the 2D harmonic oscillator. In these relationships, by replacing the universal harmonic oscillator with the q-harmonic oscillator, we obtain the energy, angular momentum, and state vector of the 2D q-hydrogen atom. That is, replacing the energy and angular momentum of the universal harmonic oscillator with those of the q-harmonic oscillator, the relations (19) and (12) give the energy and angular momentum of the 2D q-hydrogen atom, respectively, and replacing the operators a_α, $a_\alpha^\dagger$ and the state vector $|m_+, m_-\rangle''$ of the universal harmonic oscillator with the operators $a_{q\alpha}$, $a_{q\alpha}^\dagger$ and the state vector $|m_+, m_-\rangle_q''$ of the q-harmonic oscillator, the relation (19) gives the state vector of the 2D q-hydrogen atom. Hence, the energy and angular momentum of the 2D q-hydrogen atom are obtained as

$$e_{qn} = -\frac{2}{(E_{qN})^2} = -\frac{8}{\left\{ \sum_{\alpha=+,-} ([m_\alpha] + [m_\alpha + 1]) \right\}^2} \tag{45}$$

and {noting that the angular momentum operator of the 2D q-hydrogen atom is $l_{3q} = \frac{1}{2}([M_+] - [M_-])$}

$$m_q = \tfrac{1}{2} M_q = \tfrac{1}{2}([m_+] - [m_-]) \tag{46}$$

respectively, and the corresponding state vector is obtained as

$$|n, m\rangle_q = e^{-iK_{2q}\theta_{qn}} |m_+, m_-\rangle_q'' \tag{47}$$

where the operator K_{2q} is

$$K_{2q} = -\frac{i}{2} (a_{q+}^\dagger a_{q-}^\dagger - a_{q+} a_{q-}) \tag{48}$$

and the function θ_{qn} is defined by

$$\cosh \theta_{qn} = \frac{1 - 2e_{qn}}{\sqrt{-8e_{qn}}}; \qquad \sinh \theta_{qn} = -\frac{1 + 2e_{qn}}{\sqrt{-8e_{qn}}} \tag{49}$$

Here n and m may also be called the main quantum number and angular quantum number, respectively. Their connections with the quantum numbers m_+ and m_- are

$$n = \tfrac{1}{2}(m_+ + m_-) \tag{50}$$

$$m = \tfrac{1}{2}(m_+ - m_-) \tag{51}$$

For the different partitions (m_+, m_-) of N, the values of E_{qN} are not generally the same. Therefore, after the q-analogue of the hydrogen atom is made, its energy levels would be generally split.

Now let us observe in a concrete case the energy, angular momentum, and state vectors of the 2D q-hydrogen atom in its ground state and first excited state. For the ground state, $n = 0$, so $N = 0$, $(m_+, m_-) = (0, 0)$; its energy is

$$e_{q0} = -2 \tag{52}$$

which is the same as the energy of the ground state of the usual 2D hydrogen atom; its angular momentum is $m_q = 0$, and the corresponding state vector is

$$|0, 0\rangle_q = e^{-iK_{2q}\theta_{q0}}|0, 0\rangle''_q \tag{53}$$

where θ_{q0} is a function determined by

$$\cosh\theta_{q0} = \frac{1 - 2e_{q0}}{\sqrt{-8e_{q0}}}, \qquad \sinh\theta_{q0} = -\frac{1 + 2e_{q0}}{\sqrt{-8e_{q0}}} \tag{54}$$

For the first excited state, $n = 1$, so $N = 2$ and $(m_+, m_-) = (2, 0)$, $(1, 1)$, and $(0, 2)$. There are two energies corresponding to $n = 1$. One is

$$e_{q1}(1) = -\frac{2}{(1 + [2])^2} \tag{55}$$

and the corresponding state vector is

$$|1, 0\rangle_q = e^{-iK_{2q}\theta_{q1}(1)}|1, 1\rangle''_q \tag{56}$$

where $\theta_{q1}(1)$ is a function defined by

$$\cosh\theta_{q1}(1) = \frac{1 - 2e_{q1}(1)}{\sqrt{-8e_{q1}(1)}}, \qquad \sinh\theta_{q1}(1) = -\frac{1 + 2e_{q1}(1)}{\sqrt{-8e_{q1}(1)}} \tag{57}$$

In the state $|1, 0\rangle_q$, the angular momentum is $m_q = 0$. The other energy is

$$e_{q1}(2) = -\frac{8}{(1 + [2] + [3])^2} \tag{58}$$

and the corresponding state vectors are

$$|1, 1\rangle_q = e^{-iK_{2q}\theta_{q1}(2)}|2, 0\rangle''_q \tag{59}$$

and

$$|1, -1\rangle_q = e^{-iK_{2q}\theta_{q1}(2)}|0, 2\rangle''_q \tag{60}$$

where $\theta_{q1}(2)$ is a function determined by

$$\cosh\theta_{q1}(2) = \frac{1 - 2e_{q1}(2)}{\sqrt{-8e_{q1}(2)}}; \qquad \sinh\theta_{q1}(2) = -\frac{1 + 2e_{q1}(2)}{\sqrt{-8e_{q1}(2)}} \tag{61}$$

The energy $e_{q1}(2)$ is degenerate, since it corresponds to the two state vectors $|1, 1\rangle_q$ and $|1, -1\rangle_q$. In these two state vectors, the angular momenta are $m_q = \frac{1}{2}[2]$ and $m_q = -\frac{1}{2}[2]$, respectively.

The reader may write out the energies, angular momenta, and state vectors of the 2D q-hydrogen atom when $n = 2$. One can see from these observations that after the q-analogue of the hydrogen atom is made, the energy of its ground state is unchanged, while the energy levels of its excited states must be split; thus the degeneracy of these energy levels would be partly abolished. It certainly would be significant to study the physical meaning of the number q in this problem.

When $q \to 1$, the formulas (49)–(50) return to the classical case [see (12), (15), (19), (31), and (32)]. Therefore, this work is consistent.

The method presented for constructing a q-hydrogen atom in this paper is of wide significance. It is very simple and may be extended to three-, and even arbitrary-dimensional cases.

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